4-5 Notes: Integration by Substitution Name ______________________
Calculus AB

Do you remember how you did this on the 4.1-4.3 Review Worksheet?

$$\int \frac{2}{\sqrt[3]{3x}}\,dx = 2\int (3x)^{-1/3}\,dx = 2\cdot 3^{-1/3}\int x^{-1/3}dx = 2\cdot 3^{-1/3}\cdot \frac{3}{2}x^{2/3} + C = 3^{2/3}x^{2/3} + C = (3x)^{2/3} + C$$

We worked this by simplifying first, then pulled the constant out and integrated from there. There is an easier way. It is called the “change of variables approach”, but we’ll call it substitution.

## Change of Variables (Substitution)

If $u = g(x)$, then $du = g'(x)\,dx$. Therefore,

$$\int f(g(x))g'(x)dx = \int f(u)\,du = F(u) + C$$

## Guidelines for Making a Change of Variables

- Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composition functions, such as a quantity raised to a power.
- Compute $du = g'(x)\,dx$.
- Rewrite the integral in terms of the variable $u$.
- Find the resulting integral in terms of $u$.
- Replace $u$ by $g(x)$ to obtain an antiderivative in terms of $x$.
- Check your answer by differentiating.

## The General Power Rule for Integration

If $g$ is a differentiable function of $x$, then

$$\int [g(x)]^n\, g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq 1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq 1.$$

Re-do $\int \frac{2}{\sqrt[3]{3x}}\,dx$ using substitution.

**Example 1)** $\int (x^2 - 1)^3 (2x)\,dx$

**Example 2)** $\int (1 - 2x^2)^3 (-4x)\,dx$

**Example 3)** $\int x^3 \sqrt{x^4 + 2}\;dx$

**Example 4)** $\int x(4x^2 + 3)^4\,dx$

**Example 5)** $\int (5\cos 5x)\,dx$

**Example 6)** $\int \frac{x^2}{(16-x^3)^2}\,dx$

**Example 7)** $\int \frac{10x^2}{\sqrt{1+x^3}}\,dx$

**Example 8)** $\int x\sin(x^2)\,dx$

**Example 9)** $\int \sqrt{\cot x}\,\csc^2 x\,dx$

**Example 10)** $\int x\sqrt{2x+1}\,dx$

**Example 11)** $\int \frac{2x-1}{\sqrt{x+3}}\,dx$

# Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a,b]$ and $f$ is continuous on the range of $g$, then $\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$

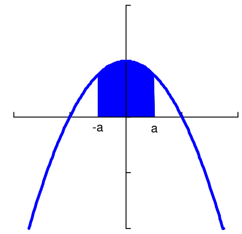
**Example 12)** $\int_0^2 \frac{x}{\sqrt{1+2x^2}}\, dx$

**Example 13)** $\int_{\pi/12}^{\pi/4} \left(\csc(2x)\cot(2x)\right) dx$

# Integration of Even and Odd Functions

Let $f$ be integrable on the closed interval $[-a, a]$.

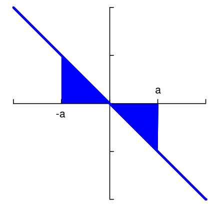
1) If $f$ is an ***even*** function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.



Even Function

- Symmetric with respect to y-axis
- Test by using: $f(-x) = f(x)$

2) If $f$ is an ***odd*** function, then $\int_{-a}^{a} f(x)\,dx = 0$.



Odd Function

- Symmetric with respect to origin
- Test by using: $f(-x) = -f(x)$

**Example 14)** $\int_{-\pi/2}^{\pi/2} \left(\sin^3 x \cos x + \sin x \cos x\right) dx$

**Example 15)** $\int_{-1}^{1} \left(x^4 + 3x^2 + 1\right) dx$

**Example 16)** $\int_{-1}^{1} \left(x^5 + 3x^3 + x\right) dx$